*J. Fluid Mech.* (1970), *vol.* 42, *part* 3, *pp.* 549–559
*Printed in Great Britain*


# Shallow three-dimensional flows with variable surface tension

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(Received 8 August 1969)

The three-dimensional steady flow of a shallow viscous liquid with non-uniform surface tension has been considered when the variation in surface tension results from the presence of an insoluble chemical contaminant on the surface. Similarly solutions for the particular problem of a channel flowing into a semi-infinite lake have been obtained, the depth and surface concentration at infinity being specified.

## 1. Introduction

There are many physical situations in which fluid motion takes place with variable surface tension, and in recent years there has been considerable interest in such phenomena; Kenning (1968) refers to a hundred publications relating to work in this field. The variation of surface tension along the interface of a fluid gives rise to tangential stresses which effect the motion of the fluid. Variation in the surface tension can occur for several reasons; examples cited by Levich (1962) are variations in the surface temperature and electric charge and changes in concentration of a surface active material.

Fluid flow with a surface active contaminant is of industrial importance and also takes place under natural conditions. A variable surface tension has probably the greatest influence on shallow flows and a two-dimensional problem of this kind has been considered by Yih (1968). In Yih's problem two reservoirs of fluid are connected by an open shallow channel with the depths of fluid and surface concentration of contaminant maintained in each reservoir. Steady motion takes place in the channel under the action of liquid head and surface tension variation.

## 2. Statement of the problem

The purpose of our paper is the extension of Yih's analysis to three-dimensional flows. A thin layer of insoluble surface active material is assumed to lie on the surface of a region of shallow liquid, the thickness of the layer being negligible compared to the depth so that it is permissible to define the concentration in terms of the density $c$ per unit area. There is no transport of contaminant into the main body of the liquid; this occurs only along the surface. The surface tension $\sigma'$ is assumed to be related linearly to the concentration, namely

$$\sigma' = \sigma'_0 + \gamma c,$$

in which $\sigma_0'$ and $\gamma$ are constants. For the purpose of our analysis it is convenient to introduce the relative surface tension $\sigma$ where

$$\sigma = \sigma' - \sigma_0',$$

and we note that $\sigma$ generally takes negative values.

The variation of concentration and hence surface tension gives rise to tractive forces along the surface which through the action of viscosity are transmitted to the bulk of the fluid. The spatial variation of hydrostatic head and surface tension will produce a steady flow of varying depth, but we shall assume that such changes are sufficiently small for the surface curvature to be neglected.

The steady state problem considered by Yih is the determination of $\sigma$ and the depth $h$ of liquid in the channel connecting the two reservoirs. Depending on the depths of the reservoirs and the direction of flow of the contaminant, two distinct situations are possible, where the bulk flow is in the direction of increasing surface tension and where it is in the direction of decreasing surface tension. In this paper we shall consider in particular the corresponding problem in which a channel of fluid flows into a semi-infinite lake, with the surface material either flowing from or into the lake depending on the relative states of contamination.

## 3. Equations of motion

With $(x, y, z)$ as Cartesian co-ordinates, $z$ is measured vertically from the horizontal bed of the liquid, which is locally of depth $h(x, y)$. If $(u, v, w)$ are Cartesian components of velocity, the diffusion equation for the surface material can be written in terms of $\sigma(=\gamma c)$, and is

$$\frac{\partial}{\partial x}(u\sigma) + \frac{\partial}{\partial y}(v\sigma) = \frac{\partial}{\partial x}\left(D\frac{\partial \sigma}{\partial x}\right) + \frac{\partial}{\partial y}\left(D\frac{\partial \sigma}{\partial y}\right) \tag{3.1}$$

at $z = h$. Here $D$ is the diffusivity of the material in the surface, and this will be assumed to be constant, as also will be the viscosity $\mu$ and density $\rho$ of the liquid.

The equations of motion of the liquid are simplified, as in the case of lubrication theory, in that inertia terms are negligible and also the dominant element only in the viscosity terms need be retained. Thus, if $p$ denotes the difference between the fluid pressure and the atmospheric pressure, the equations become

$$\left.\begin{aligned} \frac{\partial p}{\partial x} &= \mu\frac{\partial^2 u}{\partial z^2}, \\ \frac{\partial p}{\partial y} &= \mu\frac{\partial^2 v}{\partial z^2}, \\ \frac{\partial p}{\partial z} &= -\rho g, \end{aligned}\right\} \tag{3.2}$$

with the equation of continuity

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \tag{3.3}$$

Boundary conditions at $z = 0$ are

$$u = v = w = 0, \tag{3.4}$$

and at the free surface $z = h$, continuity of stress components requires that

$$\left.\begin{aligned} \mu\frac{\partial u}{\partial z} &= \frac{\partial \sigma}{\partial x}, \\ \mu\frac{\partial v}{\partial z} &= \frac{\partial \sigma}{\partial y}, \\ \text{and} \qquad p &= 0, \end{aligned}\right\} \tag{3.5}$$

in which the assumption of small surface curvature is implicit. Finally, there is the kinematical boundary condition at the free surface, namely at $z = h$,

$$w = u\frac{\partial h}{\partial x} + v\frac{\partial h}{\partial y}. \tag{3.6}$$

## 4. The field equations for $\sigma$ and $h$

Both $\sigma$ and $h$ are functions of $x$ and $y$ only, and so also are $\partial p/\partial x$, $\partial p/\partial y$ (from equations (3.2)). Thus a solution of equations (3.2) for $u$ and $v$, satisfying the boundary conditions (3.4) and (3.5), is

$$\left.\begin{aligned} u &= \frac{1}{\mu}\frac{\partial \sigma}{\partial x}z - \frac{1}{2\mu}\frac{\partial p}{\partial x}z(2h-z), \\ v &= \frac{1}{\mu}\frac{\partial \sigma}{\partial y}z - \frac{1}{2\mu}\frac{\partial p}{\partial y}z(2h-z), \end{aligned}\right\} \tag{4.1}$$

and the solution for $p$ is clearly

$$p = \rho g(h-z). \tag{4.2}$$

Levich (1962) and Yih (1968) obtained expressions similar to these for the two-dimensional channel flow problem.

With the introduction of the two-dimensional gradient operator

$$\nabla \equiv (\partial/\partial x, \partial/\partial y, 0),$$

result (4.2) yields

$$\nabla p = \rho g \nabla h,$$

and thus from (4.1), the velocity components in the surface are given by

$$(u_h, v_h) = \frac{h}{\mu}\nabla(\sigma - \tfrac{1}{4}\rho g h^2).$$

These surface values, when substituted in the diffusion equation (3.1), give the equation

$$\frac{\partial}{\partial x}\left\{\frac{h\sigma}{\mu}\frac{\partial}{\partial x}(\sigma - \tfrac{1}{4}\rho g h^2)\right\} + \frac{\partial}{\partial y}\left\{\frac{h\sigma}{\mu}\frac{\partial}{\partial y}(\sigma - \tfrac{1}{4}\rho g h^2)\right\} = D\left(\frac{\partial^2 \sigma}{\partial x^2} + \frac{\partial^2 \sigma}{\partial y^2}\right). \tag{4.3}$$

The equation of continuity (3.3), when integrated with respect to $z$ between the limits $z = 0, h$, yields with use of result (3.6),

$$\frac{\partial}{\partial x}\int_0^h u\,dz + \frac{\partial}{\partial y}\int_0^h v\,dz = 0.$$

After performing the integration with respect to $z$ of expressions (4.1), this last equation becomes

$$\frac{\partial}{\partial x}\left\{h^2\frac{\partial}{\partial x}(\sigma - \tfrac{1}{3}\rho g h^2)\right\} + \frac{\partial}{\partial y}\left\{h^2\frac{\partial}{\partial y}(\sigma - \tfrac{1}{3}\rho g h^2)\right\} = 0. \tag{4.4}$$

Equations (4.3) and (4.4) are thus the required field equations.

## 5. Similarity solution of the field equations

The field equations appear to be intractable as they stand, but it is possible to derive solutions of physical interest in the following manner. Postulate the existence of scalar fields $\phi(x,y)$, $\psi(x,y)$ defined by

$$h\sigma\nabla(\sigma - \tfrac{1}{4}\rho g h^2) - \mu D\nabla\sigma = \mu\nabla\phi, \tag{5.1}$$

$$h^2\nabla(\sigma - \tfrac{1}{3}\rho g h^2) = 2\mu\nabla\psi. \tag{5.2}$$

Equations (4.3), (4.4) imply therefore that

$$\nabla^2\phi = \nabla^2\psi = 0. \tag{5.3}$$

Further, on forming the curl of either (5.1) or (5.2) we derive the result

$$\nabla h \times \nabla\sigma = 0;$$

i.e.

$$J(\sigma, h) \equiv \frac{\partial\sigma}{\partial x}\frac{\partial h}{\partial y} - \frac{\partial\sigma}{\partial y}\frac{\partial h}{\partial x} = 0. \tag{5.4}$$

Thus $h$ and $\sigma$ are functionally related, so there exists a family of curves in the $(x,y)$ plane on each member of which $h$ and $\sigma$ assume constant values. We may thus introduce a curvilinear co-ordinate $\xi(x,y)$ so that the family is the system

$$\xi(x,y) = \text{const.},$$

and $h$ and $\sigma$ are functions of $\xi$ alone. If further we set $\phi = k_1(\xi - 1)$, $\psi = k_2(\xi - 1)$, where $k_1$, $k_2$ are constant, then $\xi$ must be harmonic, and equations (5.1), (5.2) become the ordinary differential equations

$$h\sigma\frac{d}{d\xi}(\sigma - \tfrac{1}{4}\rho g h^2) - \mu D\frac{d\sigma}{d\xi} = k_1\mu, \tag{5.5}$$

$$h^2\frac{d}{d\xi}(\sigma - \tfrac{1}{3}\rho g h^2) = 2k_2\mu, \tag{5.6}$$

and these are equivalent to the equations derived by Yih for the two-dimensional problem. We note also, from expressions (4.1), that the component of fluid velocity parallel to the base $z = 0$ is everywhere in the direction $\nabla\xi$.

The nature of the constants $k_1$, $k_2$ is not apparent from this approach, but alternatively the direct postulation of a similarity solution of (4.3), (4.4), namely $\sigma = \sigma(\xi)$, $h = h(\xi)$ leads to the conclusion that $\xi$ must be harmonic and (4.3), (4.4) then have first integrals as exhibited in (5.5), (5.6). Thus $k_1$, $k_2$ are constants associated with the surface and bulk flows respectively.

## 6. The characteristic equation

Equations (5.5) and (5.6) can be rewritten in the form

$$(h\sigma - 4\mu D)\frac{d\sigma}{d\xi} = \mu\left(4k_1 - \frac{6\sigma}{h}k_2\right), \tag{6.1}$$

$$(h\sigma - 4\mu D)\frac{dh}{d\xi} = \frac{6\mu}{\rho g h}\left\{k_1 + \frac{2k_2}{h^2}(\mu D - h\sigma)\right\}. \tag{6.2}$$

It is clear from the form of these equations that it is necessary only to consider the two cases $k_1, k_2 > 0$ and $k_1 < 0$, $k_2 > 0$. The co-ordinate $\xi$ is already dimensionless, and the following substitutions may be used to reduce equations (6.1), (6.2) to non-dimensional form:

(i) $k_1, k_2 > 0$.

$$\left.\begin{aligned} &B = 6\mu D k_1 k_2, \quad \alpha = \frac{3k_1k_2}{2DB^{\frac{1}{2}}}, \quad \beta = \frac{k_1^3}{2\rho g D B},\\ &\sigma = \frac{2B^{\frac{1}{2}}Y}{3k_2}, \quad h = \frac{B^{\frac{1}{2}}X}{k_1}. \end{aligned}\right\} \tag{6.3}$$

The above equations now become:

$$(XY-1)\frac{dY}{d\xi} = \frac{\alpha}{X}(X-Y), \tag{6.4}$$

$$(XY-1)\frac{dX}{d\xi} = \frac{\beta}{X^3}(3X^2 - 4XY + 1). \tag{6.5}$$

The characteristic, or phase-plane, equation deduced from this pair is thus

$$\frac{dY}{dX} = \frac{CX^2(X-Y)}{3X^2 - 4XY + 1}, \tag{6.6}$$

where $C = \alpha/\beta$.

(ii) $k_1 < 0$, $k_2 > 0$.

In this case the substitutions

$$\left.\begin{aligned} &B = -6\mu D k_1 k_2, \quad \alpha = \frac{3k_1k_2}{2DB^{\frac{1}{2}}}, \quad \beta = \frac{k_1^3}{2\rho g D B},\\ &\sigma = \frac{2B^{\frac{1}{2}}Y}{3k_2}, \quad h = -\frac{B^{\frac{1}{2}}X}{k_1}, \end{aligned}\right\} \tag{6.7}$$

lead to the equations

$$(XY-1)\frac{dY}{d\xi} = \frac{\alpha}{X}(X+Y), \tag{6.8}$$

$$(XY-1)\frac{dX}{d\xi} = \frac{\beta}{X^3}(3X^2 + 4XY - 1), \tag{6.9}$$

with characteristic equation

$$\frac{dY}{dX} = \frac{CX^2(X+Y)}{3X^2+4XY-1}, \tag{6.10}$$

and $C = \alpha/\beta$.

Since $h$ is essentially positive and $\sigma$ negative, the region of physical interest in both (i) and (ii) corresponds to $X > 0$, $Y < 0$. Also $C$ is positive in both cases, with $\alpha$, $\beta$ positive in (i), and negative in (ii).

## 7. Diffusion from a channel into a semi-infinite lake

The lake occupies the region $x \geqslant 0$ in the $(x, y)$ plane and the channel extends in the negative $x$ direction, its mouth being represented by the line $x = 0$, $-a \leqslant y \leqslant a$. It is supposed that $k_1, k_2 > 0$, and that the following boundary conditions apply:

$\sigma$ and $h$ have constant prescribed values on $x = 0$, $|y/a| \leqslant 1$; $\sigma \to \sigma_1$, $h \to h_1$ ($\sigma_1, h_1$ constant) as $x/a$ and $|y/a| \to \infty$; $\sigma \to \sigma_1$, $h \to h_1$ on $x = 0$, $|y/a| > 1$.

Clearly $x$, $y$ can be made non-dimensional by a simple change of variable, so that if they are now interpreted in this dimensionless form, the harmonic function $\xi$ can be set to have boundary conditions

$$\begin{aligned} &\xi = 0, \quad \text{for} \quad x = 0, \quad |y| \leqslant 1, \\ &\xi = 1, \quad \text{for} \quad x = 0, \quad |y| > 1, \\ &\xi \to 1 \quad \text{as} \quad x, \quad |y| \to \infty. \end{aligned}$$

(The region in which a solution is required is $x \geqslant 0$.)

A solution of Laplace's equation suitable for these boundary conditions can be written

$$\xi(x, y) = 1 - \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^\infty f(\lambda)\, e^{-\lambda x} \cos \lambda y \, d\lambda.$$

Thus

$$\xi(0, y) = 1 - \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^\infty f(\lambda) \cos \lambda y \, d\lambda,$$

and hence from the theory of Fourier transforms,

$$f(\lambda) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^\infty \{1 - \xi(0, y)\} \cos \lambda y \, dy = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^1 \cos \lambda y \, dy$$

$$= \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{\sin \lambda}{\lambda}.$$

Hence

$$\xi = 1 - \frac{2}{\pi} \int_0^\infty e^{-\lambda x} \frac{\sin \lambda}{\lambda} \cos \lambda y \, d\lambda,$$

and evaluation of the integral now leads to the result

$$\xi = 1 - \frac{1}{\pi} \tan^{-1}\left(\frac{2x}{x^2+y^2-1}\right). \tag{7.1}$$

The curves $\xi = \text{const.}$ are thus

$$(x + \cot(\pi\xi))^2 + y^2 = \operatorname{cosec}^2(\pi\xi), \tag{7.2}$$

which is a system of coaxal circles, with common points $(0, \pm 1)$, centres $(-\cot(\pi\xi), 0)$ and radii $\operatorname{cosec}(\pi\xi)$.

Some degree of idealization is involved in solving the problem subject to the above boundary conditions. What one would expect physically are rapid changes in contaminant concentration and liquid depth along the boundary near the mouth of the channel but that these quantities remain nearly constant thereafter. Our solution has transformed the changes into the singular points $(0, 1)$ and $(0, -1)$ at the corners of the channel. Such an assumption does in fact imply that the solution is not valid near the corners since the surface curvature will not be negligible in these regions.

We show that the boundary conditions are compatible with the requirement that the components of surface velocity at the edge of the lake shall vanish. The conditions

$$u_h = v_h = 0, \quad \text{for} \quad x = 0, \quad |y| > 1,$$

imply that

$$\xi = 1: \quad \frac{d\sigma}{d\xi} - \tfrac{1}{2}\rho g h \frac{dh}{d\xi} = 0. \tag{7.3}$$

With changes of variable (6.3), this is equivalent to

$$\xi = 1: \quad \frac{dY}{d\xi} - \tfrac{1}{4}CX\frac{dX}{d\xi} = 0. \tag{7.4}$$

The condition (7.4) must be compatible with differential equation (6.6), so that

$$\xi = 1: \quad \frac{dY}{dX} = \tfrac{1}{4}CX = \frac{CX^2(X-Y)}{3X^2 - 4XY + 1}, \tag{7.5}$$

from which it follows that $X = 1$, for all $Y$. This boundary value for $X$ has been used in subsequent numerical integrations.

## 8. Numerical results

This section refers to the solution of equations (6.4) and (6.5), but for the purpose of discussion it is convenient to refer also to the particular physical problem of §7. The trapezoidal rule, with one iteration, was used to solve the equations, and the calculations were performed on the IBM 7094 computer at Imperial College.

For given physical parameters, the ratio $k_1/k_2$ determines the dimensionless height $X$ in terms of the physical height $h$, and also the value of $C$. Equations (6.4), (6.5) were solved for $X$ and $Y$ in the range $0 \leqslant \xi \leqslant 1$, corresponding to the whole of the physical space in §7. As explained in §7 the fixed height $X = 1$ was taken at $\xi = 1$; the value of $Y$ at the same point was set at $-1$. For the solutions presented here, the value of $C$ $(= \alpha/\beta)$ was set at 1, 10, and 0·1 respectively, but $\alpha$ and $\beta$ were increased separately for each case. The corresponding graphs of $X$, $Y$ *versus* $\xi$ are displayed in figures 1, 2, and 3. From the viewpoint of the problem considered in §7, these solutions represent a situation in which the outflow from the channel has a lower concentration of surface contaminant than that existing on the lake at great distances from the channel mouth. Thus $Y$ decreases from a small negative value at $\xi = 0$ to the value $-1$ at $\xi = 1$, and in each case the decrease appears to be monotonic. Essentially, in place of prescribing $X$ and $Y$ at $\xi = 0$, the values of $\alpha$ and $\beta$ have been prescribed. Thus the solutions yield the corresponding values of $X$ and $Y$ at $\xi = 0$. In particular the computations were

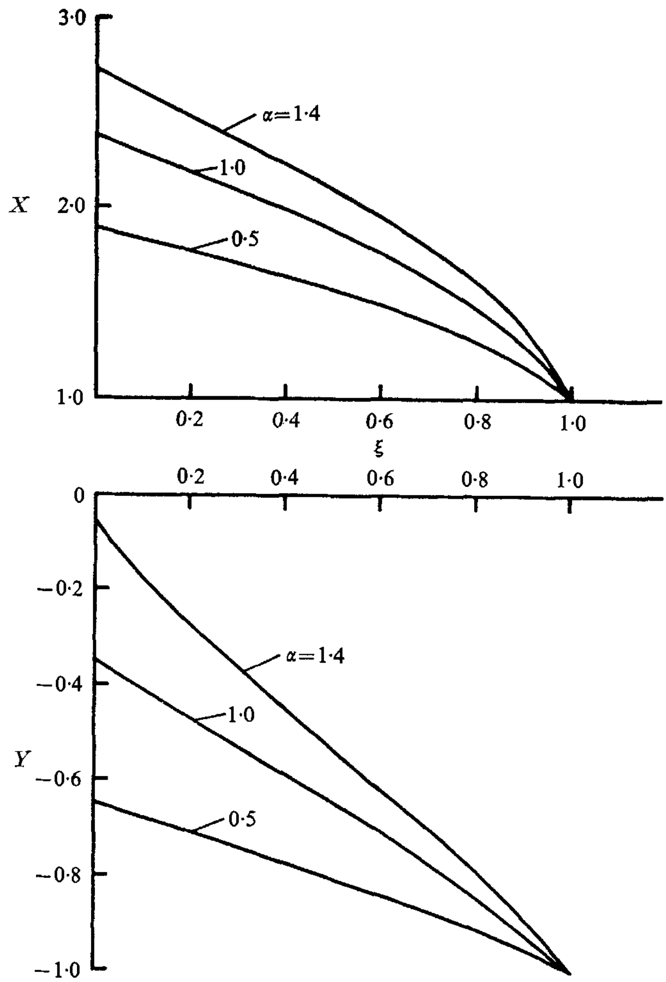


FIGURE 1. Curves of $X$ and $Y$ *versus* $\xi$ for $C = 1$.

arranged to produce, for each value of $C$, one solution for which the value of $Y$ was close to zero at $\xi = 0$. Such a solution can be seen in each of the three figures, and represents a situation in which the outflow from the channel is almost free of surface contaminant.

## 9. Discussion

The comparison of equations (5.5) and (5.6) with the corresponding equations of the two-dimensional problem shows that the constants, $k_1$, $k_2$ are associated with the surface and bulk flows respectively. In the case of flow from a channel into a semi-infinite lake they are proportional to the constant surface flux and constant bulk flux per unit width of channel. In view of the relationship between surface tension and surface concentration, the condition $k_1 > 0$ implies that surface material is flowing out of the lake and here the bulk flow is in the direction of falling surface tension. Similarly $k_1 < 0$ implies that contaminant is flowing into the lake, with the bulk flow in the direction of increasing surface tension.

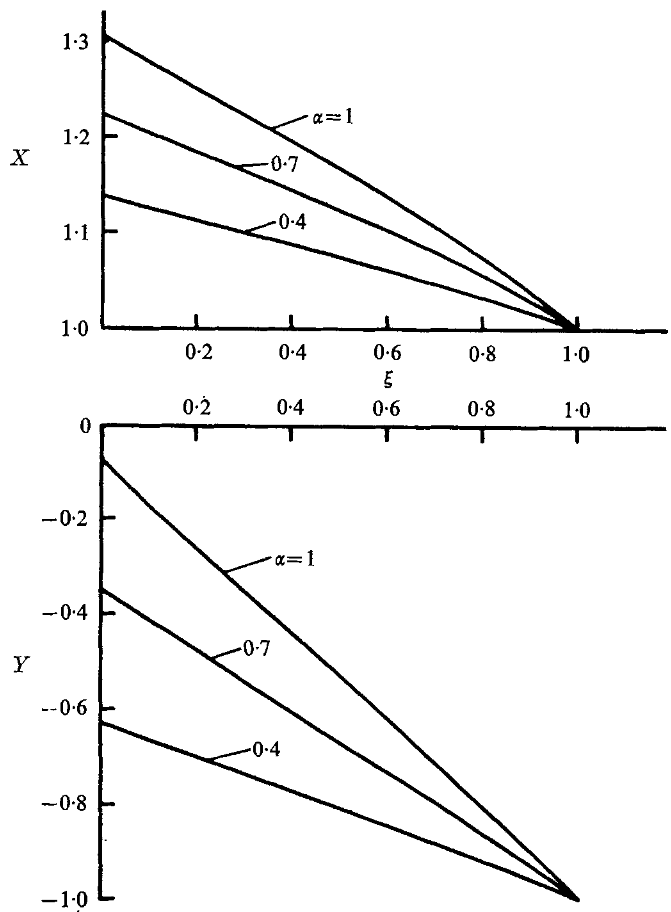


FIGURE 2. Curves of $X$ and $Y$ *versus* $\xi$ for $C = 10$.

Solutions of the flow equations have been obtained in similarity form, but such solutions will not always exist since it may not be possible to satisfy the boundary conditions. In the problem of a channel flowing into a semi-infinite lake the situation has been idealized to some extent by assuming a constant depth and constant concentration of contaminant across the mouth of the channel. The similarity solution implies that maximum changes in surface tension occur along lines of greatest slope in the surface, which appears to be a reasonable result on physical grounds.

In his analysis of the two-dimensional channel flow problem, Yih considered two physical situations—where the bulk flow is zero, and where the velocity component in the surface is zero, respectively. The solutions corresponding to zero surface flow appear to be inconsistent since the equation of continuity is not satisfied. A more serious error arises in connexion with equation (1) of Yih's paper, which is essentially the transport equation for the surface contaminant. The quantity $\sigma$ used by Yih is the relative surface tension, which for an insoluble

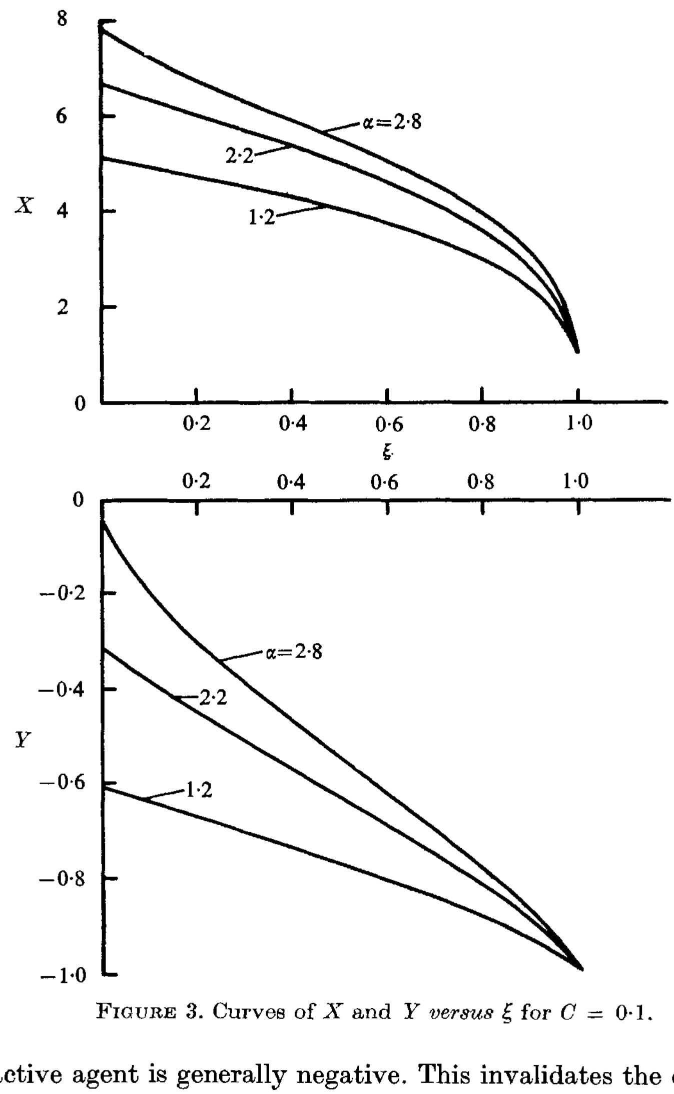


FIGURE 3. Curves of $X$ and $Y$ *versus* $\xi$ for $C = 0{\cdot}1$.

surface active agent is generally negative. This invalidates the discussion concerning the possible instability of the flow, since inequalities of the kind considered, equation (20), can no longer arise. The result that the flow is always stable may be obtained by examining the phase plane equation (6.6) of our paper, the region of physical interest being $X \geqslant 0$, $Y \leqslant 0$. Possible cusp-like solutions are associated with integral curves crossing the curve $XY = 1$. Associated with equation (6.6) are two singular points, $(1, 1)$ and $(-1, -1)$; for a certain range of the parameter $C$ a limit cycle encloses the point $(1,1)$ but this lies entirely in the region $X > 0$, $Y > 0$. It follows that flow instabilities associated with the cusp curve or with a limit cycle cannot arise.

Yih (1969) has recently considered the three-dimensional motion of a shallow liquid layer with variable surface tension, for the situation where $\Delta\sigma \gg \rho g h_0^2$, $\Delta\sigma$ being a characteristic change in $\sigma$ and $h_0$ a vertical scale. Under these conditions the flow is independent of gravity and the pressure constant throughout the fluid. He finds the depth and surface tension to be functionally related and shows that a simple polynomial of the depth is a harmonic function of the horizontal co-ordinates $x$ and $y$. The flow near vertical boundaries is dealt with by considering a velocity boundary layer whose thickness is of the same order as the depth. An explicit solution for the velocity distribution in the layer is given, for the case where the angle of contact between the free surface and the boundary is $\frac{1}{2}\pi$.

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